NON-SYMMETRIC DIFFERENTIALLY SUBORDINATE MARTINGALES AND SHARP WEAK-TYPE BOUNDS FOR FOURIER MULTIPLIERS

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Abstract. Let $p > 2$ be a given exponent. In this paper we prove, with the best constant, the weak-type (p, p) inequality

$$\|T_m f\|_{L^{p,\infty}(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)},$$

for a large class of non-symmetric Fourier multipliers T_m obtained via modulation of jumps of certain Lévy processes. In particular, the estimate holds for appropriate linear combinations of second-order Riesz transforms and skew versions of the Beurling-Ahlfors operator on the complex plane. The proof rests on a novel probabilistic bound for Hilbert-space-valued martingales satisfying a certain non-symmetric subordination principle. Further applications to harmonic functions and Riesz systems on Euclidean domains are indicated.

1. Introduction

As evidenced in numerous papers and monographs, probabilistic techniques play an important role in the study of boundedness of various objects in harmonic analysis, often offering sharp or at least tight results. The purpose of this paper is to explore further this interesting direction and shows how a certain fine-tuned estimate for martingales leads to sharp weak-type bound for a wide class of Fourier multipliers.

We start with recalling the necessary background and notations which will be used in our considerations below. For any bounded function $m : \mathbb{R}^d \to \mathbb{C}$, there exists a bounded linear operator T_m on $L^2(\mathbb{R}^d)$, called the Fourier multiplier with the symbol m, given by the identity $\widehat{T_m f} = m\widehat{f}$ involving the corresponding Fourier transforms. By Plancherel's theorem, the norm of T_m on $L^2(\mathbb{R}^d)$ is equal to $\|m\|_\infty$ and it has been long of interest to investigate those symbols m, for which the associated multipliers extend to bounded operators on (all or some) $L^p(\mathbb{R}^d)$, or some other function spaces. In our considerations below, we will investigate tight weak-type estimates for a class of symbols which have a nice probabilistic interpretation. They are called Lévy symbols in the literature, their study was started with the paper [8] and continued in several works (see e.g. [9, 10, 13, 26] and consult the references therein). To recall these, we need some additional notations. Let ν be a Lévy measure on $\mathbb{R}^d$, i.e., a nonnegative Borel measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min\{|x|^2, 1\}\nu(\mathrm{d}x) < \infty.$$

Let μ be a finite Borel measure on the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$ and fix two Borel functions ϕ on $\mathbb{R}^d$ and ψ on $\mathbb{S}$ with values in $\mathbb{R}$. We define the associated multiplier $m = m_{\phi,\psi,\mu,\nu}$ on $\mathbb{R}^d$ by

$$m(\xi) = \frac{\frac{1}{2}\int_{\mathbb{S}} \langle \xi, \theta\rangle^2 \psi(\theta)\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\phi(x)\nu(\mathrm{d}x)}{\frac{1}{2}\int_{\mathbb{S}} \langle \xi, \theta\rangle^2 \mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\nu(\mathrm{d}x)} \tag{1.1}$$

if the denominator is not 0, and $m(\xi) = 0$ otherwise. Here $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^d$.

Many important examples are included in this class, as we show now. Let $e_1, e_2, \ldots, e_d$ be the collection of unit vectors in $\mathbb{R}^d$ and let δ_{e_j} be the Dirac measure concentrated on e_j. If we

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take $\nu = 0$, $\mu = \delta_{e_1} + \delta_{e_2} + \ldots \delta_{e_d}$ and consider ψ which is equal to 1 on e_j and vanishes for all other e_k's, then $m(\xi) = \frac{\xi_j^2}{|\xi|^2}$, i.e., T_m is the second-order Riesz transform R_j^2, an absolutely classical object in harmonic analysis and potential theory (cf. [40]). This in turn leads us to another crucial example in the planar case $d = 2$. Recall that Beurling-Ahlfors operator can be defined as the Fourier multiplier with the symbol $m(\xi) = (\xi/|\xi|)^2$, $\xi \in \mathbb{C} \setminus \{0\}$ (with the standard identification $\mathbb{C} \simeq \mathbb{R}^2$). Alternatively, we can define $\mathcal{B}$ by the identity

$$\mathcal{B} = \mathcal{R}_1^2 - \mathcal{R}_2^2 + 2i\mathcal{R}_1\mathcal{R}_2$$

, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the Riesz transforms in $\mathbb{R}^2$. This operator plays a fundamental role in the study of quasi-conformal mappings, partial differential equations and complex analysis: we refer the interested reader to the monograph [4] for the detailed exposition of the subject. In particular, evaluating the precise L^p norm of $\mathcal{B}$ is a long-standing open problem; a celebrated conjecture of T. Iwaniec [27] states that

$$\|\mathcal{B}\|_{L^p(\mathbb{C})\to L^p(\mathbb{C})} = \max\{p-1, (p-1)^{-1}\}, \qquad 1 < p < \infty,$$

and its validity or failure would have many profound consequences (see e.g. [4, 5]). Analogous questions about the boundedness of R_j^2 or $\mathcal{B}$ on other function spaces have also been studied intensively in the literature (see e.g. [25, 28, 29]) and applied to regularity of solutions to certain classes of elliptic PDEs and certain aspects of geometric function theory.

The list of meaningful examples of multipliers with symbols (1.1) is much longer. We present some of them, following the exposition in [7]. Let $\mu \equiv 0$ and let ν be the Lévy measure of a non-zero symmetric α-stable Lévy process in $\mathbb{R}^d$, $\alpha \in (0, 2)$ (for the relevant definitions, we refer the reader to [39]). In polar coordinates, we have the identity

$$\nu(\mathrm{d}r\mathrm{d}\theta) = r^{-1-\alpha}\mathrm{d}r\sigma(\mathrm{d}\theta), \qquad r > 0,\, \theta \in \mathbb{S},$$

where the so-called spectral measure σ is finite and non-zero on $\mathbb{S}$. Pick a function $\phi : \mathbb{R}^d \to [0, 1]$ homogeneous of order 0, that is, satisfying $\phi(x) = \phi(x/|x|)$ for $x \neq 0$. Let $c_\alpha = \int_0^\infty [1 - \cos s]s^{-1-\alpha}\mathrm{d}s$. We have

$$\begin{aligned}\int_{\mathbb{R}^d} [1 - \cos\langle\xi, x\rangle]\phi(x)\nu(\mathrm{d}x) &= \int_{\mathbb{S}}\int_0^\infty [1 - \cos\langle\xi, r\theta\rangle]\phi(r\theta)r^{-1-\alpha}\mathrm{d}r\sigma(\mathrm{d}\theta) \\ &= c_\alpha \int_{\mathbb{S}} |\langle\xi, \theta\rangle|^\alpha \phi(\theta)\sigma(\mathrm{d}\theta)\end{aligned}$$

which leads us to the symbol

$$M_\alpha(\xi) = \frac{\int_{\mathbb{S}} |\langle\xi, \theta\rangle|^\alpha \phi(\theta)\sigma(\mathrm{d}\theta)}{\int_{\mathbb{S}} |\langle\xi, \theta\rangle|^\alpha \sigma(\mathrm{d}\theta)}. \tag{1.2}$$

In particular, if we take σ to be the probability measure satisfying $\sigma(e_k) = 1/d$ for each k and ϕ is the indicator function of the j-th axis, we obtain Marcinkiewicz-type multipliers (see Stein [40], p. 110):

$$M_{\alpha,j}(\xi) = \frac{|\xi_j|^\alpha}{|\xi_1|^\alpha + |\xi_2|^\alpha + \ldots + |\xi_d|^\alpha}. \tag{1.3}$$

Note that if we pass with α to 2, we obtain the second-order Riesz transforms R_j^2. To present another example, assume that d is even: $d = 2n$, and let σ be the uniform measure on the set

$$\{x \in \mathbb{S} : x_1^2 + \ldots + x_n^2 = 1 \text{ or } x_{n+1}^2 + x_{n+2}^2 + \ldots + x_{2n}^2 = 1\}.$$

If ϕ is the indicator function of $\{x \in \mathbb{S} : x_1^2 + \ldots + x_n^2 = 1\}$, then (1.2) becomes

$$M(\xi) = \frac{|\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2|^{\alpha/2}}{|\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2|^{\alpha/2} + |\xi_{n+1}^2 + \xi_{n+2}^2 + \ldots + \xi_{2n}^2|^{\alpha/2}}.$$

Our final example is related to the class of the so-called tempered stable Lévy processes [42]. Set $\mu \equiv 0$ and define the Lévy measure ν in polar coordinates by

$$\nu(\mathrm{d}r\mathrm{d}\theta) = r^{-1-\alpha}\mathrm{d}r\sigma(\mathrm{d}\theta), \qquad r > 0,\, \theta \in \mathbb{S},$$

where σ is the general spectral measure as above. This yields the multiplier

$$M(\xi)=\frac{\int_{\mathbb{S}}\log[1+\langle\xi,\theta\rangle^2]\phi(\theta)\sigma(\mathrm{d}\theta)}{\int_{\mathbb{S}}\log[1+\langle\xi,\theta\rangle^2]\sigma(\mathrm{d}\theta)}.$$

In particular, by choosing ϕ, σ as previously, we get the logarithmic multipliers

$$M_j(\xi)=\frac{\log(1+\xi_j^2)}{\log(1+\xi_1^2)+\log(1+\xi_2^2)+\ldots+\log(1+\xi_d^2)},\quad j=1,\,2,\,\ldots,\,d.$$

The above examples justify the interest in various tight inequalities for the wider class of Fourier multipliers, with symbols given by (1.1). This subject has been investigated in many papers and extended beyond the Euclidean setting [1, 11]. For example, the seminal paper [8] contains the proof of the estimate

$$\|T_m\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}\leq \max\{p-1,(p-1)^{-1}\},\qquad 1<p<\infty,$$

under the assumption that ϕ and ψ take values in $[-1,1]$. Then it was proved in [13, 26] that the estimate is sharp, for $T_m=\mathcal{R}_1^2-\mathcal{R}_2^2$. The corresponding sharp weak-type, logarithmic, exponential and restricted estimates can be found in [34, 37, 38].

It is natural to ask about the sharp versions of the above estimates if we restrict the range of the functions ϕ and ψ to some interval different than $[-1,1]$. For example, as we have seen above, the second-order Riesz transform R_j^2 of the Marcinkiewicz multiplier (1.3) are obtained with $[0,1]$-valued functions. In our considerations below, we will assume that the functions ϕ and ψ take values in the interval $[b,B]$, where b and B are fixed parameters satisfying $b\leq 0<B$ and $b+B>0$ (the latter condition can always be imposed, due to the symmetry). For instance, the "asymmetric" second-order Riesz transform

$$T_m=BR_j^2+bR_k^2,$$

or, more generally, the "skew" Marcinkiewicz multiplier with the symbol

$$M_{\alpha,j,k}(\xi)=\frac{B|\xi_j|^\alpha+b|\xi_k|^\alpha}{|\xi_1|^\alpha+|\xi_2|^\alpha+\ldots+|\xi_d|^\alpha},$$

$j\neq k$, can be obtained in such a manner. To the best of our knowledge, the question about the best constant $c_{p,b,B}$ in the L^p estimate

$$\|T_mf\|_{L^p(\mathbb{R}^d)}\leq c_{p,b,B}\|f\|_{L^p(\mathbb{R}^d)},\qquad 1<p<\infty,$$

is still open (the paper [10] relates its value to a sharp constant in a certain martingale inequality). In the very special case $b=0$ and $B=1$, its description is contained in Choi's paper [21] and is quite technical.

The contribution of our present paper is the identification of the constant in the corresponding weak-type estimate. To state the result, we need an auxiliary parameter. Given $p>2$ and b, B as above, let $c=c_{p,b,B}$ be the unique number in $(1,\infty)$ satisfying

$$(B-b)c^{p-1}=2Bc+B+b. \tag{1.4}$$

(The existence and uniqueness follow easily from Darboux property, the strict convexity of the function $c\mapsto c^{p-1}$ and the fact that for $c=1$, the left-hand side is smaller than the right-hand side).

Theorem 1.1. *Let μ, ν, ϕ and ψ be as above and let m be defined by (1.1). Then the multiplier T_m satisfies*

$$\|T_mf\|_{L^{p,\infty}(\mathbb{R}^d)}\leq C_{p,b,B}\|f\|_{L^p(\mathbb{R}^d)},\qquad 2<p<\infty, \tag{1.5}$$

where

$$C_{p,b,B}=\frac{1}{2}\left[\frac{(2Bc+(p-1)(B-b))^{p-1}(B-b)}{c+1}\right]^{1/p}. \tag{1.6}$$

The constant is the best possible for each dimension $d\geq 2$.

We would like to emphasize that the above result holds in the range $p > 2$ only: for remaining values of p (i.e., for $1 \leq p \leq 2$) we have been unable to push the calculations through. Nevertheless, we believe that the above implicit description of the best constant $C_{p,b,B}$ is very nice and relatively simple.

Actually, we will show that the estimate (1.5) is sharp in the special case of non-symmetric second-order Riesz transforms, i.e., for $T_m = BR_1^2 + bR_2^2$. Inequalities for such operators are of interest from the viewpoint of regularity of solutions to certain partial differential equations and potential theory. Consider the following simple application, motivated by the discussion in [40, p. 59-60]. Suppose that f is of class C^2 on $\mathbb{R}^d$ and has compact support. Then we have the sharp estimate

$$\left\| \frac{B\partial^2 f}{\partial x_1^2} + \frac{b\partial^2 f}{\partial x_2^2} \right\|_{L^{p,\infty}(\mathbb{R}^d)} \leq C_{p,b,B} \|\Delta f\|_{L^p(\mathbb{R}^d)}.$$

A few words about our approach and the organization of the paper are in order. The proof will rest on an appropriate novel sharp inequality for martingales satisfying a certain non-symmetric subordination, which is of independent interest and connections. This probabilistic result will be established in the next section, actually, in a full range $1 \leq p < \infty$ (but with a non-sharp constant when $1 \leq p < 2$). Section 3 is devoted to the proof of Theorem 1.1. The final part of the paper contains some further applications of the martingales estimates in the study of harmonic functions on Euclidean domains.

2. New martingale inequalities

The probabilistic contribution of the paper is also significant on its own. We present quite a detailed presentation. For convenience, the contents of this section is split into several separate parts.

2.1. Definitions and the statement of the results. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$ be an adapted discrete-time martingale taking values in some separable Hilbert space $\mathcal{H}$; the norm and the scalar product in this space will be denoted by $|\cdot|$ and $\cdot$, respectively. Without loss of generality, we may and do assume that the space is equal to ℓ^2. Assume further that $(df_n)_{n\geq 0}$ stands for the difference sequence of f, i.e.,

$$df_0 = f_0 \quad \text{and} \quad df_n = f_n - f_{n-1} \quad \text{for all } n \geq 1.$$

Let g be a transform of f by a predictable sequence $v = (v_n)_{n\geq 0}$ bounded in absolute value by 1: that is, we have $dg_n = v_n df_n$ for all n and each term v_n is measurable with respect to $\mathcal{F}_{(n-1)\vee 0}$. Then, following Burkholder [17], for $1 < p < \infty$ we have the sharp strong-type inequality

$$\|g\|_p \leq (p^* - 1)\|f\|_p \quad (\text{where} \quad p^* = \max\{p, p/(p-1)\}). \tag{2.1}$$

Here we have used the notation $\|f\|_p = \sup_n \|f_n\|_p$. In the boundary case $p = 1$ the above moment inequality does not hold with any finite constant, but one can establish the corresponding weak-type bound. If $1 \leq p \leq 2$, then ([17])

$$\|g\|_{p,\infty} \leq \left(\frac{2}{\Gamma(p+1)}\right)^{1/p} \|f\|_p, \tag{2.2}$$

where $\|g\|_{p,\infty} = \sup_{\lambda>0} \lambda \mathbb{P}(\sup_n |g_n| \geq \lambda)^{1/p}$. For $p > 2$, Suh [43] showed that

$$\|g\|_{p,\infty} \leq \left(\frac{p^{p-1}}{2}\right)^{1/p} \|f\|_p. \tag{2.3}$$

Both (2.2), (2.3) are sharp. These inequalities (i.e, strong or weak type) have also been studied in the less restrictive setting in which the martingale g is assumed to be *differentially subordinate* to f. The latter means that for each $n \geq 0$ we have the almost sure bound $|dg_n| \leq |df_n|$. One can also extend the strong- and weak-type estimates for martingale transforms in other directions. For instance, one can consider the case in which the transforming sequence $(v_n)_{n\geq 0}$ takes values in $[0, 1]$: see Burkholder [18] and Choi [21]. In analogy to the previous case, one can investigate the more general context invoking the appropriately modified, non-symmetric differential subordination, which reads

$$|dg_n|^2 \leq df_n \cdot dg_n, \quad n = 0, 1, 2, \dots. \tag{2.4}$$

This is equivalent to saying that the martingale $-\frac{f}{2} + g$ is differentially subordinate to $\frac{f}{2}$. Under this assumption, we have the strong type estimate [21] (see also [35]),

$$\|g\|_p \leq c_p \|f\|_p. \tag{2.5}$$

The best constant c_p is quite complicated, we refer the reader to the above papers for the precise description. We also have the following sharp weak type bound [36]: if $1 \leq p \leq 2$, then

$$\|g\|_{p,\infty} \leq \|f\|_p \tag{2.6}$$

while for $p > 2$, we have

$$\|g\|_{p,\infty} \leq C_p \|f\|_p, \tag{2.7}$$

where $C_p = \frac{1}{2}\left[\frac{(2c+p-1)^{p-1}}{c+1}\right]^{1/p}$ and $c = c(p) > 1$ is the unique positive number satisfying $c^{p-1} = 2c + 1$.

In our considerations below, we will establish the weak-type (p, p) bound in a more general setting. Namely, we consider the case of martingale transforms in which the sequence $(v_n)_{n \geq 0}$ takes values in the interval $[b, B]$ for some $b \leq 0 < B$ satisfying $b + B > 0$. As previously, we will study the more general subordinate context: namely, we assume that martingales f and g are such that $g - \frac{b+B}{2} f$ is differentially subordinate to $\frac{B-b}{2} f$. It is easy to check that this non-symmetric version of differential subordination does generalize the previous setup of martingale transforms. To the best of our knowledge, the best constant $c_{p,b,B}$ in the L^p estimate

$$\|g\|_p \leq c_{p,b,B} \|f\|_p \tag{2.8}$$

seems to be unkown, while we will prove the following related result, i.e., the weak-type (p, p) estimate.

Theorem 2.1. *If f, g are two Hilbert-space-valued martingales satisfying the following non-symmetric condition*

$$g - \frac{B+b}{2} f \quad \textit{is differentially subordinate to} \quad \frac{B-b}{2} f,$$

then for all $1 \leq p < \infty$ we have

$$\|g\|_{p,\infty} \leq C_{p,b,B} \|f\|_p.$$

Here $C_{p,b,B} = B - b$ for $1 \leq p \leq 2$, and $C_{p,b,B}$ is given by (1.6) for $p > 2$. In the latter case, the constant is the best possible. It is already optimal if $\mathcal{H} = \mathbb{R}$ and g is assumed to be the transform of f by a predictable sequence taking values in the interval $[b, B]$.

Unfortunately, we have been unable to prove the sharp version of (2.10) in the range $1 \leq p \leq 2$. We easily see that the above result does generalize (2.7).

We would like to point out that all the above estimates hold also in the more general continuous-time case. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t \geq 0}$ of sub-σ-fields of $\mathcal{F}$. We assume in addition that $\mathcal{F}_0$ contains all the events of probability 0. Suppose further that X, Y are two adapted martingales taking values in $\mathcal{H} = \ell^2$. As usual, we impose standard regularity restrictions on trajectories of X and Y: we assume that the paths are right-continuous and have limits from the left. Then $[X, X]$, the quadratic covariance process of X, is given by $[X, X] = \sum_{n=1}^{\infty} [X^n, X^n]$, where X^n is the n-th coordinate of X and $[X^n, X^n]$ is the usual square bracket of the real-valued martingale X^n (see Dellacherie and Meyer [23] for details). Following Bañuelos and Wang [12] and Wang [45], we say that the martingale Y is differentially subordinate to X, if the process $([X, X]_t - [Y, Y]_t)_{t \geq 0}$ is nondecreasing and nonnegative as a function of t. Treating two given discrete-time martingales f, g as continuous-time processes (via $X_t = f_{\lfloor t \rfloor}$ and $Y_t = g_{\lfloor t \rfloor}$, $t \geq 0$), we see this new domination is consistent with the original definition discussed previously.

The continuous-time generalization of Theorem 2.1 is as follows. In analogy to the discrete-time case, we use the notation $\|X\|_p = \sup_{t\geq 0}\|X_t\|_p$ and $\|Y\|_{p,\infty} = \sup_{t\geq 0}\|Y_t\|_{p,\infty}$ for the strong and weak p-th norms.

Theorem 2.2. *If X, Y are two Hilbert-space-valued martingales such that*

$$Y - \frac{B+b}{2}X \quad \textit{is differentially subordinate to} \quad \frac{B-b}{2}X, \tag{2.9}$$

then for all $1 \leq p < \infty$ we have

$$\|Y\|_{p,\infty} \leq C_{p,b,B}\|X\|_p. \tag{2.10}$$

For $p > 2$, the constant $C_{p,b,B}$ given by (1.6) is the best possible.

The inequality (2.10) will be proved with the so-called Burkholder's method (sometimes also referred to as the Bellman function technique). More specifically, we will deduce the validity of the estimate from the existence of a certain special function, enjoying appropriate size and concavity requirement. This argument goes back to the seminal works [17, 19] of Burkholder and has been extended in many directions. For an exhaustive presentation of the method, its connections to the theory of PDEs, and numerous examples, we refer the interested reader to the monograph [35].

2.2. **Proof of Theorem 2.2, the case $1 \leq p \leq 2$.** In this range of the parameter p, the proof is relatively simple. We contain the details for a convenience. Define the functions $\mathcal{U}_p$ and $\mathcal{V}_p$ by the formulas

$$\mathcal{U}_p(x,y) = \begin{cases} py\cdot(y-(B+b)x) + pbB|x|^2 & \text{if } \frac{B-b}{2}|x| + |y - \frac{B+b}{2}x| < 1 \\ p - p(B-b)|x| & \text{otherwise} \end{cases}$$

and

$$\mathcal{V}_p(x,y) = \chi_{\{|y|\geq 1\}} - (B-b)^p|x|^p.$$

We have the following majorization.

Lemma 2.3. *For all $x, y \in \mathcal{H}$, we have*

$$\mathcal{U}_p(x,y) \geq \mathcal{V}_p(x,y). \tag{2.11}$$

Proof. If $|y| \geq 1$, $\mathcal{V}_p(x,y) = 1 - (B-b)^p|x|^p$ and

$$1 \leq |y| \leq \left|y - \frac{B+b}{2}x\right| + \frac{B+b}{2}|x| \leq \left|y - \frac{B+b}{2}x\right| + \frac{B-b}{2}|x|,$$

then $\mathcal{U}_p(x,y) = p - p(B-b)|x| \geq 1 - (B-b)^p|x|^p = \mathcal{V}_p(x,y)$ since $p - ps \geq 1 - s^p$, for all $s \geq 0$, by virtue of the mean-value theorem. So, suppose that $|y| < 1$. We consider two cases. If $|2y - (B+b)x| + (B-b)|x| \geq 2$, then

$$\mathcal{U}_p(x,y) = p - p(B-b)|x| \geq 1 - (B-b)^p|x|^p > (B-b)^p|x|^p = \mathcal{V}_p(x,y)$$

and we are done. Finally, if $|2y - (B+b)x| + (B-b)|x| < 2$, then we have

$$\mathcal{V}_p(x,y) = -(B-b)^p|x|^p \leq p\frac{-(B-b)^2}{2}|x|^2 \leq p\frac{-(B-b)^2}{4}|x|^2$$

(by the mean value property of the convex function $t \mapsto t^{p/2}$) and hence

$$\begin{aligned}\mathcal{V}_p(x,y) - \mathcal{U}_p(x,y) &\leq -p|y|^2 + p(B+b)x\cdot y - p\left(\frac{B+b}{2}\right)^2|x|^2 \\ &= -p\left|y - \frac{B+b}{2}x\right|^2 \leq 0.\end{aligned}$$

The proof is complete. □

We are ready to establish the weak-type estimate.

Proof of (2.10). Consider an auxiliary special function $u_1 : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ given by

$$u_1(x,y) = \begin{cases} |y|^2 - |x|^2 & \text{if } |x| + |y| < 1, \\ 1 - 2|x| & \text{if } |x| + |y| \geq 1. \end{cases}$$

Theorem 5.6 in [35] asserts that if $\xi = (\xi_t)_{t\geq 0}$, $\zeta = (\zeta_t)_{t\geq 0}$ are continuous-time $\mathcal{H}$-valued martingales such that ζ is differentially subordinate to ξ, then $\mathbb{E}u_1(\xi_t, \zeta_t) \leq 0$ for all t. Applying this estimate to $\xi = \frac{B-b}{2}X$ and $\zeta = Y - \frac{B+b}{2}X$ and noting that for all x, y

$$\mathcal{U}_p(x,y) = pu_1\left(\frac{B-b}{2}x, y - \frac{B+b}{2}x\right),$$

we obtain $\mathbb{E}\mathcal{U}_p(X_t, Y_t) \leq 0$. Combining this with the previous lemma, we conclude that $\mathbb{E}\mathcal{V}_p(X_t, Y_t) \leq 0$ and hence

$$\mathbb{P}(|Y_t| \geq 1) \leq (B-b)^p \|X_t\|_p^p \leq (B-b)^p \|X\|_p^p.$$

The non-symmetric subordination of Y to X is preserved if we divide both processes by a positive number, so we get

$$\|Y_t\|_{p,\infty}^p = \sup_{\lambda > 0} \lambda^p \mathbb{P}(|Y_t| \geq \lambda) \leq (B-b)^p \|X\|_p^p$$

and taking the supremum over all t yields the desired assertion. □

Remark 2.4. Using a standard stopping time argument, one can establish a slightly stronger inequality

$$\left\| \sup_{t\geq 0} |Y_t| \right\|_{p,\infty} \leq (B-b)\|X\|_p,$$

which involves the maximal function of Y on the left. To see this, fix $0 < \lambda' < \lambda$, introduce the stopping time $\tau = \inf\{t : |Y_t| \geq \lambda'\}$ (with the convention $\inf \emptyset = +\infty$) and note that the nonsymmetric differential subordination is preserved if we pass to the stopped martingales $(X_{\tau\wedge t})_{t\geq 0}$ and $(Y_{\tau\wedge t})_{t\geq 0}$. Consequently,

$$(\lambda')^p \mathbb{P}(|Y_{\tau\wedge t}| \geq \lambda') \leq (B-b)^p \mathbb{E}|X_{\tau\wedge t}|^p \leq (B-b)^p \mathbb{E}|X_t|^p \leq (B-b)^p \|X\|_p^p.$$

It remains to note that $\{\sup_{t\geq 0} |Y_t| \geq \lambda\} \subseteq \bigcup_{t\geq 0}\{|g_{\tau\wedge t}| \geq \lambda'\}$ and the event appearing in the union are nondecreasing. Hence,

$$(\lambda')^p \mathbb{P}(\sup_{t\geq 0} |Y_{\tau\wedge t}| \geq \lambda) \leq (B-b)^p \|X\|_p^p$$

and letting $\lambda' \uparrow \lambda$ completes the proof, since $\lambda > 0$ was arbitrary.

2.3. **Proof of** (2.10) **for** $p > 2$**.** Here the analysis will be much more involved. We would also like to emphasize that the reasoning will be a significant improvement of that appearing in [36]: actually, we will study a slightly stronger form of the weak-type estimate which will enable us later to pass to the corresponding sharp inequality for Fourier multipliers.

We start with the definition of an auxiliary function $\mathcal{U}_\infty : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, given by the formula

$$\mathcal{U}_\infty(x,y) = \begin{cases} 0 & \text{if } \frac{B-b}{2}|x| + |y - \frac{B+b}{2}x| \leq 1, \\ \left(\left|y - \frac{B+b}{2}x\right| - 1\right)^2 - \left(\frac{B-b}{2}\right)^2 |x|^2 & \text{if } \frac{B-b}{2}|x| + |y - \frac{B+b}{2}x| > 1. \end{cases}$$

Next, recall $c \in (1,\infty)$ given by (1.4) and set

$$D = D_{p,b,B} = \frac{(p-1)(B-b)}{(p-1)(B-b) + 2Bc}.$$

We have $0 < D < 1 - p^{-1}$; the first estimate is clear and the second follows easily from the inequality $c > 1$. The special function $\mathcal{U}_p : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ corresponding to the weak-type estimate is defined by

$$\mathcal{U}_p(x,y) = \frac{2^{p-1}p(p-2)C_{p,b,B}^p}{(p-1)^{p-2}(B-b)^p} \int_0^D s^{p-1} \mathcal{U}_\infty\left(\frac{x}{s}, \frac{y}{s}\right) \mathrm{d}s.$$

After some lengthy, but direct and rather straightforward calculations we get

$$\mathcal{U}_p(x,y) = \frac{2^p C_{p,b,B}^p}{(p-1)^{p-1}(B-b)^p}\left(\frac{B-b}{2}|x| + \left|y - \frac{B+b}{2}x\right|\right)^{p-1}$$
$$\times\left(\left|y - \frac{B+b}{2}x\right| + (1-p)\frac{B-b}{2}|x|\right)$$

if $\frac{B-b}{2}|x| + \left|y - \frac{B+b}{2}x\right| \leq D$, and

$$\mathcal{U}_p(x,y) = \frac{p(p-1)(p-2)}{2(c+1)}\left(\frac{|y-\frac{B+b}{2}x|^2 - (\frac{B-b}{2})^2|x|^2}{(p-2)D} - \frac{2|y-\frac{B+b}{2}x|}{p-1} + \frac{D}{p}\right)$$

otherwise. Finally, define the function $\mathcal{V}_p : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ by the formula

$$\mathcal{V}_p(x,y) = p(|y| - 1 + p^{-1})_+ - C_{p,b,B}^p|x|^p.$$

We will show that $\mathcal{U}_p \geq \mathcal{V}_p$ in the following three lemmas. First, notice that it suffices to establish the majorization in the real case and for x, y satisfying $0 \leq x \leq \frac{2}{B+b}y$. Indeed, let us (for a moment) write $\mathcal{U}_p^{\mathcal{H}}$, $\mathcal{V}_p^{\mathcal{H}}$ instead of $\mathcal{U}_p$, $\mathcal{V}_p$, to indicate the Hilbert space we are working with. For $x, y \in \mathcal{H}$, take $x' = |x|$ and $y' = |y - \frac{B+b}{2}x| + \frac{B+b}{2}|x|$. Then

$$0 \leq x' \leq \frac{2}{B+b}y', y' - \frac{B+b}{2}x' = |y - \frac{B+b}{2}x|$$

and $y' \geq |y|$, so

$$\mathcal{U}_p^{\mathcal{H}}(x,y) = \mathcal{U}_p^{\mathbb{R}}(x',y') \quad \text{and} \quad \mathcal{V}_p^{\mathcal{H}}(x,y) \leq \mathcal{V}_p^{\mathbb{R}}(x',y').$$

Consequently, we have

$$\mathcal{U}_p^{\mathcal{H}}(x,y) - \mathcal{V}_p^{\mathcal{H}}(x,y) \geq \mathcal{U}_p^{\mathbb{R}}(x',y') - \mathcal{V}_p^{\mathbb{R}}(x',y')$$

and hence it is enough to work under the additional assumptions on x and y formulated above.

Lemma 2.5. *We have $\mathcal{U}_p(x,y) \geq \mathcal{V}_p(x,y)$ for $y - bx \leq D$.*

Proof. The assumption $y - bx \leq D$ is equivalent (under the above conditions on x and y) to $\frac{B-b}{2}|x| + \left|y - \frac{B+b}{2}x\right| \leq D$. Since $D \leq 1 - p^{-1}$, the assertion can be rewritten in the form

$$\frac{2^p}{(p-1)^{p-1}(B-b)^p}(y-bx)^{p-1}\left(y + x\cdot\frac{b(p-2)-pB}{2}\right) \geq -x^p.$$

By continuity, we may restrict ourselves to $x > 0$. If we divide both sides by x^p, the desired estimate becomes $F_0(\frac{y}{x} - b) \geq 0$, where

$$F_0(s) = \frac{2^p}{(p-1)^{p-1}(B-b)^p}s^{p-1}\left(s - p\frac{B-b}{2}\right) + 1, \qquad s > 0.$$

Note that $F_0'(s) \geq 0$ if $s \geq \frac{B-b}{2}(p-1)$, $F_0'(s) < 0$ for $s < \frac{B-b}{2}(p-1)$ and

$$F_0'(\frac{B-b}{2}(p-1)) = F_0(\frac{B-b}{2}(p-1)) = 0.$$

This implies $F_0(s) \geq 0$ for all s and we are done. $\square$

Lemma 2.6. *We have $\mathcal{U}_p(x,y) \geq \mathcal{V}_p(x,y)$ for $bx + D < y < 1 - p^{-1}$.*

Proof. Since $y > bx + D$, we have

$$\frac{B-b}{2}|x| + \left|y - \frac{B+b}{2}x\right| > D$$

and hence the majorization is equivalent to

$$\frac{p(p-1)(p-2)}{2(c+1)}\left(\frac{y^2 + bBx^2 - (B+b)xy}{(p-2)D} - \frac{2(y - \frac{B+b}{2}x)}{p-1} + \frac{D}{p}\right) + C_{p,b,B}^p x^p \geq 0.$$

Denote the left side by $F_1(y)$. Then $F_1'(y) \geq 0$ if and only if $y \geq \frac{B+b}{2}x + D\frac{p-2}{p-1}$, so we will be done if we show that

$$\frac{p(p-1)(p-2)}{2(c+1)}\left(\frac{-(\frac{B-b}{2}x)^2}{(p-2)D} + \frac{D}{p(p-1)^2}\right) + C_{p,b,B}^p x^p \geq 0. \tag{2.12}$$

The left hand side of (2.12) is equal to $F_2\left(\left(\frac{B-b}{2}x\right)^2\right)$, where F_2 is given by

$$F_2(s) = \frac{p(p-1)(p-2)}{2(c+1)}\left(-\frac{s}{(p-2)D} + \frac{D}{p(p-1)^2}\right) + C_{p,b,B}^p \left(\frac{2}{B-b}\right)^p s^{p/2}.$$

It suffices to note that F_2 is convex and

$$F_2'\left(\left(\frac{D}{p-1}\right)^2\right) = F_2\left(\left(\frac{D}{p-1}\right)^2\right) = 0. \tag{2.13}$$

The proof is complete. □

Lemma 2.7. *We have $\mathcal{U}_p(x,y) \geq \mathcal{V}_p(x,y)$ for $y \geq 1 - p^{-1}$.*

Proof. The estimate is equivalent to

$$\frac{p(p-1)(p-2)}{2(c+1)}\left\{\frac{y^2 + bBx^2 - (B+b)xy}{(p-2)D} - \frac{2(y - \frac{B+b}{2}x)}{p-1} + \frac{D}{p}\right\} \\ - p(y - 1 + p^{-1}) + C_{p,b,B}^p x^p \geq 0.$$

We will show the validity of the above bound for all x, $y \geq 0$. Denote the left hand side by $F_3(x,y)$. The partial derivative of F_3 with respect to y is given by

$$\begin{aligned} F_{3y}(x,y) &= \frac{p(p-1)(p-2)}{2(c+1)}\left\{\frac{2y - (B+b)x}{(p-2)D} - \frac{2}{p-1}\right\} - p \\ &= \frac{p(p-1)}{(c+1)D}\left\{y - \frac{B+b}{2}x - 1 + \frac{(B+b)(1-D)}{2B}\right\}. \end{aligned}$$

Thus, it suffices to establish the estimate on the halfline

$$y = \frac{B+b}{2}x + 1 - \frac{(B+b)(1-D)}{2B}.$$

For such x, y, the claim becomes

$$-\frac{p(p-1)}{2(c+1)D}\left(\frac{B-b}{2}x\right)^2 - \frac{p(B+b)}{2}x + C_{p,b,B}^p x^p + \xi \geq 0,$$

where ξ does not depend on x. Denoting the left-hand side by $F_4(x)$, we check that

$$F_4\left(\frac{1-D}{B}\right) = F_4'\left(\frac{1-D}{B}\right) = 0.$$

However, it is clear that $F_4'(0) < 0$ and there is a constant $\eta > 0$ such that F_4 is concave on $[0, \eta)$ and convex on (η, ∞). This proves that $F_4(x) \geq 0$, which completes the proof of the lemma. □

We are ready for the proof of the weak-type bound.

Proof of (2.10). We may assume that X is L^p-bounded, since otherwise there is nothing to prove. Then Y is also L^p-bounded, by Burkholder's moment estimate (2.1) (or rather its continuous-time version established by Wang [45]). Indeed,

$$\|Y\|_p \leq \left\|Y - \frac{B+b}{2}X\right\|_p + \frac{B+b}{2}\|X\|_p \leq (p^*-1)\frac{B-b}{2}\|X\|_p + \frac{B+b}{2}\|X\|_p < \infty.$$

As in the case $p \leq 2$, the key argument is to consider an auxiliary special function. Let $u_\infty : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be given by

$$u_\infty(x,y) = \begin{cases} 0 & \text{if } |x|+|y| \leq 1, \\ (|y|-1)^2 - |x|^2 & \text{if } |x|+|y| > 1 \end{cases}$$

By Theorem 5.6 in [35], if $\xi = (\xi_t)_{t\geq 0}$, $\zeta = (\zeta_t)_{t\geq 0}$ is a pair of continuous-time, $\mathcal{H}$-valued and L^2-bounded martingales such that ζ is differentially subordinate to ξ, then $\mathbb{E}u_\infty(\xi_t, \zeta_t) \leq 0$ for all $t \geq 0$. We apply this bound to the pair $\xi = \frac{B-b}{2}X$ and $\zeta = Y - \frac{B+b}{2}X$ and observe that

$$\mathcal{U}_\infty(x,y) = u_\infty\left(\frac{B-b}{2}x, y - \frac{B+b}{2}x\right)$$

for all x, y; consequently, we obtain $\mathbb{E}\mathcal{U}_\infty(X_t, Y_t) \leq 0$. By Fubini's theorem, this immediately yields $\mathbb{E}\mathcal{U}_p(X_t, Y_t) \leq 0$ for all t. (To see that Fubini's theorem is applicable, notice that $|\mathcal{U}_\infty(x,y)| \leq K(|x|^2 + |y|^2 + 1)$ for some constant K and therefore,

$$\mathbb{E}\int_0^D ps^{p-1}|\mathcal{U}_\infty(X_t/s, Y_t/s)|\mathrm{d}s \leq K'\mathbb{E}(|X_t|^2 + |Y_t|^2 + 1) < \infty,$$

for some K'.) Now, using the majorization $\mathcal{U}_p \geq \mathcal{V}_p$, we obtain

$$p\mathbb{E}(|Y_t| - 1 + p^{-1})_+ \leq C_{p,b,B}^p \mathbb{E}|X_t|^p. \tag{2.14}$$

It remains to note that $\mathbb{P}(|Y_t| \geq 1) \leq p\mathbb{E}(|Y_t| - 1 + p^{-1})_+$, and the proof is completed exactly in the same manner as in the case $p \leq 2$. We would also like to mention here that Remark 2.4 also applies here and yields the stronger estimate

$$\left\| \sup_{t\geq 0} |Y_t| \right\|_{p,\infty} \leq C_{p,b,B}\|X\|_p. \quad \square$$

Remark 2.8. The reason why we consider the stronger (but a little strange-looking) estimate (2.14) is that the expression on the left is a convex function of Y_t. This will be crucial for our applications in Section 3: roughly speaking, Fourier multipliers we consider there can be represented as conditional expectations of certain dominated martingales Y, and the efficient study of these requires the use of Jensen's inequality.

2.4. **Sharpness for** $p > 2$**.** Now we will prove that the constant $C_{p,b,B}$ cannot be improved, even if we restrict ourselves to the discrete-time and real-valued case, in which the martingale g is assumed to be the transform of f by a predictable sequence with values in $\{b, B\}$. Let $p > 2$ be fixed.

Let N be a huge positive integer, let $\varepsilon < D$ be a small positive number and let $\delta > 0$ be determined by the condition

$$\varepsilon(1 + (B-b)\delta)^N = D, \tag{2.15}$$

that is, $\delta = (B-b)^{-1}\left((D/\varepsilon)^{1/N} - 1\right)$. Finally, consider the auxiliary parameter $d = \frac{2}{(B-b)(p-1)}$.

Consider the Markov martingale (f,g) with the distribution uniquely determined by the following requirements.

(i) The process starts from $(0, \varepsilon)$.
(ii) The state of the form $(0,\gamma)$, $\gamma < D$, leads to $(-d\gamma, \gamma - Bd\gamma)$ or to $(\delta\gamma, \gamma + B\delta\gamma)$.
(iii) The state of the form $(\delta\gamma, \gamma + B\delta\gamma)$ leads to $(0, \gamma + (B-b)\delta\gamma)$ or to $(d\gamma, \gamma + (B-b)\delta\gamma + bd\gamma)$.
(iv) The state $(0, D)$ leads to $(-dD, D - BdD)$ or to $(\frac{1-D}{B}, 1)$.
(v) All other states are absorbing.

We do not need to specify the transition probabilities in (ii), (iii) and (iv), they are determined by the requirement that (f,g) is a martingale. It is not difficult to see that $g - \varepsilon$ is a transform of f by the predictable sequence taking values in $\{b, B\}$: specifically, the transforming sequence $v = (v_n)_{n\geq 0}$ is equal to b for even n and B for odd n. Note that the pair (f,g) terminates after at most $2N + 1$ steps.

Let us study the distributions of f and g. Directly from the above construction, we see that

$$\mathbb{P}(g_{2N+1}-\varepsilon\geq 1-\varepsilon)=\mathbb{P}(g_{2N+1}\geq 1)=\left(\frac{d-\delta}{d+\delta}\right)^N\cdot\frac{Dd}{Dd+\frac{1-D}{B}}.$$

For f, the calculations are more involved. The absolute value of the terminal variable, $|f_{2N+1}|$, is concentrated on the set

$$\left\{d\varepsilon,(1+(B-b)\delta)d\varepsilon,(1+(B-b)\delta)^2d\varepsilon\dots,(1+(B-b)\delta)^Nd\varepsilon,\frac{1-D}{B}\right\}.$$

The corresponding probabilities are the following. For $n=0,\,1,\,2,\,\dots,\,N-1$,

$$\mathbb{P}(|f_{2N+1}|=(1+(B-b)\delta)^nd\varepsilon)=\left(\frac{d-\delta}{d+\delta}\right)^n\cdot\frac{2\delta}{d+\delta}.$$

In addition,

$$\mathbb{P}(|f_{2N+1}|=(1+(B-b)\delta)^Nd\varepsilon)=\mathbb{P}(|f_{2N+1}|=Dd)=\left(\frac{d-\delta}{d+\delta}\right)^N\cdot\frac{\frac{1-D}{B}}{Dd+\frac{1-D}{B}}$$

and

$$\mathbb{P}\left(|f_{2N+1}|=\frac{1-D}{B}\right)=\left(\frac{d-\delta}{d+\delta}\right)^N\cdot\frac{Dd}{Dd+\frac{1-D}{B}}.$$

Consequently, we have

$$\begin{aligned}\mathbb{E}|f_{2N+1}|^p&=\sum_{n=0}^{N-1}[(1+(B-b)\delta)^nd\varepsilon]^p\cdot\left(\frac{d-\delta}{d+\delta}\right)^n\cdot\frac{2\delta}{d+\delta}\\&\quad+\left(\frac{d-\delta}{d+\delta}\right)^N\left[(Dd)^p\cdot\frac{\frac{1-D}{B}}{Dd+\frac{1-D}{B}}+\left(\frac{1-D}{B}\right)^p\cdot\frac{Dd}{Dd+\frac{1-D}{B}}\right]\\&=\frac{\varepsilon^pd^p\cdot 2\delta}{d+\delta}\cdot\frac{(1+(B-b)\delta)^{Np}\left(\frac{d-\delta}{d+\delta}\right)^N-1}{(1+(B-b)\delta)^p\left(\frac{d-\delta}{d+\delta}\right)-1}\\&\quad+\left(\frac{d-\delta}{d+\delta}\right)^N\left[(Dd)^p\cdot\frac{\frac{1-D}{B}}{Dd+\frac{1-D}{B}}+\left(\frac{1-D}{B}\right)^p\cdot\frac{Dd}{Dd+\frac{1-D}{B}}\right].\end{aligned}$$

Now, if we let $N\to\infty$, then $\delta\to 0$; more precisely, (2.15) implies that $N\delta\to(B-b)^{-1}\log(D/\varepsilon)$. Consequently, we see that

$$\left(\frac{d-\delta}{d+\delta}\right)^N\to(\varepsilon/D)^{2/(d(B-b))}=(\varepsilon/D)^{p-1}$$

and

$$((1+(B-b)\delta)^p\left(\frac{d-\delta}{d+\delta}\right)-1)/\delta\to p(B-b)-\frac{2}{d}=B-b.$$

Putting all these observations together, we see that

$$\mathbb{P}(g_{2N+1}-\varepsilon\geq 1-\varepsilon)\to(\varepsilon/D)^{p-1}\cdot\frac{Dd}{Dd+\frac{1-D}{B}},$$

furthermore, $\mathbb{E}|f_{2N+1}|^p$ converges to

$$\frac{2\varepsilon^pd^{p-1}\left(\frac{D}{\varepsilon}-1\right)}{B-b}+\left(\frac{\varepsilon}{D}\right)^{p-1}\left[(Dd)^p\cdot\frac{\frac{1-D}{B}}{Dd+\frac{1-D}{B}}+\left(\frac{1-D}{B}\right)^p\cdot\frac{Dd}{Dd+\frac{1-D}{B}}\right].$$

Consequently, the ratio $\mathbb{E}|f_{2N+1}|^p/\mathbb{P}(|g_{2N+1}-\varepsilon|\geq 1-\varepsilon)$ approaches

$$\frac{2(Dd)^{p-2}\,(D-\varepsilon)}{B-b}\cdot\left(Dd+\frac{1-D}{B}\right)+\left[(Dd)^{p-1}\cdot\frac{1-D}{B}+\left(\frac{1-D}{B}\right)^p\right].$$

It remains to observe that ε was arbitrary: sending it to zero, we check that the above expression converges to $C_{p,b,B}^{-p}$. Indeed, dividing both sides by $(Dd)^{p-1}$, the latter is equivalent to

$$\frac{2}{B-b}\left(D+\frac{1-D}{Bd}\right)+\frac{1-D}{B}+Dd\left(\frac{1-D}{BDd}\right)^p=C_{p,b,B}^{-p}D^{1-p}d^{1-p}.$$

Now, directly from the definitions of $C_{p,b,B}$, D and d, the right-hand side is equal to $2(c+1)/(B-b)$. Furthermore, we have $c=(1-D)/(BDd)$, so if we multiply both sides by $(B-b)/D$, the above equality becomes $2(c+1)+(B-b)(dc+dc^p)=\frac{2(c+1)}{D}$, or

$$\frac{B-b}{B}(c+c^p)=2(c+1)\cdot\frac{1-D}{BDd}.$$

But $(1-D)/(BDd)=c$, so if we divide both sides by c, we see that the desired identity is equivalent to (1.4). This establishes the desired sharpness.

3. A weak-type inequality for Fourier multipliers

Now we will see how the probabilistic estimates established in the previous section lead to corresponding estimates for Fourier multipliers. We split the contents into separate subsections.

3.1. **Stochastic representation of the multipliers from the class (1.1).** This is described in full detail in [8], so we shall be brief. Let m be the multiplier as in (1.1), with the corresponding parameters ϕ, ψ, μ and ν. Assume in addition that $\nu(\mathbb{R}^d)$ is finite and nonzero, and put $\tilde{\nu}=\nu/|\nu|$. Consider the independent random variables T_{-1}, T_{-2}, $\ldots$, Z_{-1}, Z_{-2}, $\ldots$ such that for each $n=-1,-2,\ldots$, T_n has exponential distribution with parameter $|\nu|$, while Z_n takes values in $\mathbb{R}^d$ and has $\tilde{\nu}$ as the distribution. Next, put $S_n=-(T_{-1}+T_{-2}+\ldots+T_n)$ for $n=-1,-2,\ldots$ and introduce the family of compound Poisson processes

$$X_{s,t}=\sum_{s<S_j\leq t}Z_j,\qquad X_{s,t-}=\sum_{s<S_j<t}Z_j,\qquad \Delta X_{s,t}=X_{s,t}-X_{s,t-},$$

for $-\infty<s\leq t\leq 0$. Next, for a given sufficiently regular function $f:\mathbb{R}^d\to\mathbb{C}^n$ (say, belonging to C_0^∞), define the corresponding parabolic extension $\mathcal{U}_f$ to $(-\infty,0]\times\mathbb{R}^d$ by the formula

$$\mathcal{U}_f(s,x)=\mathbb{E}f(x+X_{s,0}).$$

For a fixed $x\in\mathbb{R}^d$, $s<0$, $f:\mathbb{R}^d\to\mathbb{C}^n$ and $\phi:\mathbb{R}^d\to[b,B]$, we consider the processes $F=(F_t^{x,s,f})_{s\leq t\leq 0}$ and $G=(G_t^{x,s,f,\phi})_{s\leq t\leq 0}$ given by

$$\begin{aligned} F_t&=\mathcal{U}_f(t,x+X_{s,t}),\\ G_t&=\sum_{s<u\leq t}\big[(F_u-F_{u-})\cdot\phi(X_{s,u}-X_{s,u-})\big]\\ &\quad-\int_s^t\int_{\mathbb{R}^d}\big[\mathcal{U}_f(v,x+X_{s,v-}+z)-\mathcal{U}_f(v,x+X_{s,v-})\big]\phi(z)\nu(\mathrm{d}z)\mathrm{d}v. \end{aligned}\tag{3.1}$$

These processes enjoy the following properties.

Lemma 3.1. *For any fixed x, s, f, ϕ as above, the processes $F^{x,s,f}$, $G^{x,s,f,\phi}$ are $\mathbb{C}^n$-valued martingales with respect to $(\mathcal{F}_t)_{s\leq t\leq 0}=(\sigma(X_{s,t}:s\leq t))_{s\leq t\leq 0}$. Furthermore, $G^{x,s,f,\phi}-\frac{B+b}{2}F^{x,s,f}$ is differentially subordinate to $\frac{B-b}{2}F^{x,s,f}$.*

Proof. The fact that F and G are martingales was proved in [8]. Actually, F is a pure-jump martingale and G is obtained by modulating the jumps of F (by means of the function ϕ) and subtracting the appropriate compensator which ensures that the martingale property is preserved; note that G is also pure-jump. Consequently, the square brackets of these processes are just sums of squares of appropriate jumps, and hence

$$\left[G-\frac{B+b}{2}F,G-\frac{B+b}{2}F\right]_t-\left[\frac{B-b}{2}F,\frac{B-b}{2}F\right]_t$$
$$=\sum_{s\le u<t}(F_u-F_{u-})^2\left\{\left(\phi(X_{s,u}-X_{s,u-})-\frac{B+b}{2}\right)^2-\left(\frac{B-b}{2}\right)^2\right\}.$$

Since ϕ takes values in $[b,B]$, the expression in the parentheses is nonpositive and hence the differential subordination follows. □

The final step is to define the operator $\mathcal{S}=\mathcal{S}^{s,\phi,\nu}$ by the bilinear form

$$\int_{\mathbb{R}^d}\langle\mathcal{S}f(x),g(x)\rangle\mathrm{d}x=\int_{\mathbb{R}^d}\mathbb{E}\big\langle G_0^{x,s,f,\phi},g(x+X_{s,0})\big\rangle\mathrm{d}x, \tag{3.2}$$

where $f,\,g\in C_0^\infty(\mathbb{R}^d)$. We have the following statement, proved in [8] and [36], which establishes the aforementioned representation of Fourier multipliers in terms of Lévy processes.

Lemma 3.2. *Let $1<p<\infty$ and $d\ge 2$. The operator $\mathcal{S}^{s,\phi,\nu}$ is well defined and extends to a bounded operator on $L^p(\mathbb{R}^d)$, which can be expressed as a Fourier multiplier with the symbol*

$$M(\xi)=M_{s,\phi,\nu}(\xi)$$
$$=\left[1-\exp\left(2s\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz)\right)\right]\frac{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\phi(z)\nu(dz)}{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz)}$$

if $\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz)\neq 0$, and $M(\xi)=0$ otherwise.

3.2. **Proof of (1.5).** We may and do assume that at least one of the measures μ, ν is nonzero. It is convenient to split the reasoning into two parts.

Step 1. First we show the estimate for the multipliers of the form

$$M_{\phi,\nu}(\xi)=\frac{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\phi(z)\nu(\mathrm{d}z)}{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(\mathrm{d}z)}. \tag{3.3}$$

Assume that $0<\nu(\mathbb{R}^d)<\infty$; then the above representation in terms of Lévy processes is applicable. Fix $s<0$ and functions $f,\,g\in C_0^\infty(\mathbb{R}^d)$ such that f takes values in $\mathbb{C}^n$, while g takes values in the unit ball of $\mathbb{C}^n$ and is supported on a certain set E of finite Lebesgue measure. By Fubini's theorem and (2.14), we have, for any $\lambda>0$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mathbb{E}\big[G_0^{x,s,f,\phi}g(x+X_{s,0})\big]\mathrm{d}x\\
&\le\int_{\mathbb{R}^d}\mathbb{E}|G_0^{x,s,f,\phi}|1_{\{x+X_{s,0}\in E\}}\mathrm{d}x\\
&\le\int_{\mathbb{R}^d}\mathbb{E}(|G_0^{x,s,f,\phi}|-\lambda)1_{\{x+X_{s,0}\in E\}}\mathrm{d}x+\lambda\int_{\mathbb{R}^d}\mathbb{E}1_{\{x+X_{s,0}\in E\}}\mathrm{d}x\\
&\le\int_{\mathbb{R}^d}\mathbb{E}(|G_0^{x,s,f,\phi}|-\lambda)_+\mathrm{d}x+\lambda|E|\\
&=\frac{\lambda}{1-\frac1p}\int_{\mathbb{R}^d}\mathbb{E}\left(\frac{1-\frac1p}{\lambda}|G_0^{x,s,f,\phi}|-1+\frac1p\right)_+\mathrm{d}x+\lambda|E|\\
&\le\frac{\lambda}{1-\frac1p}\cdot\frac{C_{p,b,B}^p}{p}\int_{\mathbb{R}^d}\mathbb{E}\left(\frac{1-\frac1p}{\lambda}|F_0^{x,s,f}|\right)^p\mathrm{d}x+\lambda|E|\\
&=\frac{\left(1-\frac1p\right)^{p-1}}{p\lambda^{p-1}}\cdot C_{p,b,B}^p\|f\|_{L^p(\mathbb{R}^d)}^p+\lambda|E|.
\end{aligned}$$

Plugging this into the definition of $\mathcal{S}$ and taking the supremum over all g as above, we obtain

$$\int_E |\mathcal{S}^{s,\phi,\nu} f(x)| \mathrm{d}x \leq \frac{\left(1-\frac{1}{p}\right)^{p-1}}{p\lambda^{p-1}} \cdot C_{p,b,B}^p \|f\|_{L^p(\mathbb{R}^d)}^p + \lambda |E|. \tag{3.4}$$

Now if we let $s \to -\infty$, then $M_{s,\phi,\nu}$ converges pointwise to the multiplier $M_{\phi,\nu}$ given by (3.3). By Plancherel's theorem, $\mathcal{S}^{s,\phi,\nu} f \to T_{M_{\phi,\nu}} f$ in $L^2(\mathbb{R}^d)$ and hence there is a sequence $(s_n)_{n=1}^\infty$ converging to $-\infty$ such that $\lim_{n\to\infty} \mathcal{S}^{s_n,\phi,\nu} f \to T_{M_{\phi,\nu}} f$ almost everywhere. Thus Fatou's lemma combined with (3.4) yields the bound

$$\int_E |T_{M_{\phi,\nu}} f(x)| \mathrm{d}x \leq \frac{\left(1-\frac{1}{p}\right)^{p-1}}{p\lambda^{p-1}} \cdot C_{p,b,B}^p \|f\|_{L^p(\mathbb{R}^d)}^p + \lambda |E|.$$

Now we minimize the right-hand side over λ. A straightforward analysis of the derivative shows that the minimum is attained for $\lambda = (p-1) C_{p,b,B} \|f\|_{L^p(\mathbb{R}^d)} / (p|E|^{1/p})$, and we obtain the estimate

$$\int_E |T_{M_{\phi,\nu}} f(x)| \mathrm{d}x \leq C_{p,b,B} \|f\|_{L^p(\mathbb{R}^d)} |E|^{(p-1)/p}. \tag{3.5}$$

Now, if we fix an arbitrary $\lambda > 0$ and set $E = \{|T_{M_{\phi,\nu}} f(x)| \geq \lambda\}$, then

$$\lambda |E|^{1/p} \leq \frac{1}{|E|^{(p-1)/p}} \int_E |T_{M_{\phi,\nu}} f(x)| \mathrm{d}x \leq C_{p,b,B} \|f\|_{L^p(\mathbb{R}^d)}, \tag{3.6}$$

which is the desired weak-type bound.

Step 2. Now we deduce the result for the general multipliers as in (1.1) and drop the assumption $0 < \nu(\mathbb{R}^d) < \infty$. For a given $\varepsilon > 0$, define a Lévy measure ν_ε in polar coordinates $(r, \theta) \in (0, \infty) \times \mathbb{S}$ by

$$\nu_\varepsilon(\mathrm{d}r \mathrm{d}\theta) = \varepsilon^{-2} \delta_\varepsilon(\mathrm{d}r) \mu(d\theta).$$

Here δ_ε denotes Dirac measure on $\{\varepsilon\}$. Next, consider a multiplier $M_{\varepsilon,\phi,\psi,\mu,\nu}$ as in (3.3), in which the Lévy measure is $1_{\{|x|>\varepsilon\}}\nu + \nu_\varepsilon$ and the jump modulator is given by $1_{\{|x|>\varepsilon\}}\phi(x) + 1_{\{|x|=\varepsilon\}}\psi(x/|x|)$. Note that this Lévy measure is finite and nonzero, at least for sufficiently small ε. If we let $\varepsilon \to 0$, we see that

$$\begin{aligned} \int_{\mathbb{R}^d} [1 - \cos\langle \xi, x\rangle] \psi(x/|x|) \nu_\varepsilon(\mathrm{d}x) &= \int_{\mathbb{S}} \langle \xi, \theta\rangle^2 \phi(\theta) \frac{1 - \cos\langle \xi, \varepsilon\theta\rangle}{\langle \xi, \varepsilon\theta\rangle^2} \mu(d\theta) \\ &\to \frac{1}{2} \int_{\mathbb{S}} \langle \xi, \theta\rangle^2 \phi(\theta) \mu(\mathrm{d}\theta) \end{aligned}$$

and, consequently, $M_{\varepsilon,\phi,\psi,\mu,\nu} \to m_{\phi,\psi,\mu,\nu}$ pointwise. Therefore, by Fatou's lemma, (3.5) yields

$$\int_E |T_{m_{\phi,\psi,\mu,\nu}} f(x)| \mathrm{d}x \leq C_{p,b,B} \|f\|_{L^p(\mathbb{R}^d)} |E|^{(p-1)/p}.$$

This in turn gives (1.5), by the same procedure as in (3.6) above.

3.3. **Sharpness.** Now we will prove that the constant $C_{p,b,B}$ is optimal, for each $p > 2$ and any dimension $d \geq 2$. We start with a simple observation which follows from the construction in Section 2 and a simple scaling argument. Fix an arbitrary $\varepsilon > 0$ and a continuous function $\Theta : [0, \infty) \to [0, 1]$ such that $\Theta(x) = 0$ for $x \in [0, 1]$ and $\Theta(x) = 1$ for $x > 1 + \varepsilon$. Then there is a pair (F, G) of finite martingales such that G is a transform of F by a predictable sequence with values in $\{b, B\}$ and

$$\mathbb{E}\Theta(|G_\infty|) > (C_{p,b,B}^p - \varepsilon) \mathbb{E}|F_\infty|^p. \tag{3.7}$$

We will find an appropriate analytic analogue of this estimate, with the expectation replaced by an integral over $\mathbb{R}^d$, F_∞ replaced with a certain function f and G_∞ replaced with $T_m f$ for an appropriate symbol m. This will be done with the use of laminates, important family of probability measures on matrices. It is convenient to split the reasoning into several separate parts. For the sake of convenience and to make the presentation as self contained as possible, we recall the preliminaries on laminates and their connections to martingales from [13] and [38], Section 4.2.

Laminates. Assume that $\mathbb{R}^{m\times n}$ stands for the space of all real matrices of dimension $m\times n$ and $\mathbb{R}^{n\times n}_{sym}$ denote the subclass of $\mathbb{R}^{n\times n}$ which consists of all symmetric matrices of dimension $n\times n$.

Definition 3.3. A function $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ is called *rank-one convex*, if for all $A,B\in\mathbb{R}^{m\times n}$ with rank $B=1$, the function $t\mapsto f(A+tB)$ is convex.

See [22, p. 100] for other equivalent definitions of rank-one convexity. Suppose that $\mathcal{P}=\mathcal{P}(\mathbb{R}^{m\times n})$ is the class of all compactly supported probability measures on $\mathbb{R}^{m\times n}$. For a measure $\nu\in\mathcal{P}$, we define

$$\overline{\nu}=\int_{\mathbb{R}^{m\times n}}Xd\nu(X),$$

the associated *center of mass* or *barycenter* of ν.

Definition 3.4. We say that a measure $\nu\in\mathcal{P}$ is a *laminate*, if

$$f(\overline{\nu})\leq\int_{\mathbb{R}^{m\times n}}f\mathrm{d}\nu$$

for all rank-one convex functions f. We will write $\nu\in\mathcal{L}$ in such a case. The set of laminates with barycenter 0 is denoted by $\mathcal{L}_0(\mathbb{R}^{m\times n})$.

Laminates can be used to obtain lower bounds for solutions of certain PDEs, as observed by Faraco in [24]. In addition, laminates appear naturally in the context of convex integration, where they lead to interesting counterexamples, see e.g. [3], [20], [31], [33] and [44]. For our results here we will be interested in the case of 2×2 symmetric matrices. The key observation is that laminates can be regarded as probability measures that record the distribution of the gradients of smooth maps: see Corollary 3.8 below. We briefly explain this and refer the reader to the works [30], [33] and [44] for full details.

Definition 3.5. Let U be a subset of $\mathbb{R}^{2\times 2}$ and let $\mathcal{PL}(U)$ denote the smallest class of probability measures on U which

(i) contains all measures of the form $\lambda\delta_A+(1-\lambda)\delta_B$ with $\lambda\in[0,1]$ and satisfying $\mathrm{rank}(A-B)=1$;
(ii) is closed under splitting in the following sense: if $\lambda\delta_A+(1-\lambda)\nu$ belongs to $\mathcal{PL}(U)$ for some $\nu\in\mathcal{P}(\mathbb{R}^{2\times 2})$ and μ also belongs to $\mathcal{PL}(U)$ with $\overline{\mu}=A$, then also $\lambda\mu+(1-\lambda)\nu$ belongs to $\mathcal{PL}(U)$.

The class $\mathcal{PL}(U)$ is called the *prelaminates* in U.

It follows immediately from the definition that the class $\mathcal{PL}(U)$ only contains atomic measures. Also, by a successive application of Jensen's inequality, we have the inclusion $\mathcal{PL}\subset\mathcal{L}$. The following are two well known lemmas in the theory of laminates; see [3], [30], [33], [44].

Lemma 3.6. *Let $\nu=\sum_{i=1}^N\lambda_i\delta_{A_i}\in\mathcal{PL}(\mathbb{R}^{2\times 2}_{sym})$ with $\overline{\nu}=0$. Moreover, let $0<r<\frac{1}{2}\min|A_i-A_j|$ and $\delta>0$. For any bounded domain $\mathcal{B}\subset\mathbb{R}^2$ there exists $u\in W_0^{2,\infty}(\mathcal{B})$ such that $\|u\|_{C^1}<\delta$ and for all $i=1,2,\ldots,N$,*

$$\big|\{x\in\mathcal{B}:\,|D^2u(x)-A_i|<r\}\big|=\lambda_i|\mathcal{B}|.$$

Lemma 3.7. *Let $K\subset\mathbb{R}^{2\times 2}_{sym}$ be a compact convex set and suppose that $\nu\in\mathcal{L}(\mathbb{R}^{2\times 2}_{sym})$ satisfies $\mathrm{supp}\,\nu\subset K$. For any relatively open set $U\subset\mathbb{R}^{2\times 2}_{sym}$ with $K\subset U$, there exists a sequence $\nu_j\in\mathcal{PL}(U)$ of prelaminates with $\overline{\nu}_j=\overline{\nu}$ and $\nu_j\overset{*}{\rightharpoonup}\nu$, where $\overset{*}{\rightharpoonup}$ denotes weak convergence of measures.*

Combining these two lemmas and using a simple mollification, we obtain the following statement, proved by Boros, Shékelyhidi Jr. and Volberg [13]. It exhibits the connection between laminates supported on symmetric matrices and second derivatives of functions. It will be our main tool in the proof of the sharpness. Recall that $\mathbb{D}$ denotes the unit disc of $\mathbb{C}$.

Corollary 3.8. *Let $\nu \in \mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})$. Then there exists a sequence $u_j \in C_0^\infty(\mathbb{D})$ with uniformly bounded second derivatives, such that*

$$\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \varphi(D^2 u_j(x))\,dx \;\to\; \int_{\mathbb{R}^{2\times 2}_{sym}} \varphi\,d\nu$$

for all continuous $\varphi : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}$.

Biconvex functions and a special laminate. The next step in our analysis is devoted to the introduction of a certain special laminate. We need some additional notation. A function $\zeta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is said to be *biconvex* if for any fixed $z \in \mathbb{R}$, the functions $x \mapsto \zeta(x,z)$ and $y \mapsto \zeta(z,y)$ are convex. Now, take the martingales F and G as in (3.7). Then the "skew" martingale pair

$$(\widetilde{F},\widetilde{G}) := \left(\frac{-bF+G}{B-b}, \frac{BF-G}{B-b}\right)$$

is finite, starts from $(0,0)$ and has the following *zigzag* property: for any $n \geq 0$ we have $\widetilde{F}_n = \widetilde{F}_{n+1}$ with probability 1 or $\widetilde{G}_n = \widetilde{G}_{n+1}$ almost surely; that is, in each step $(\widetilde{F},\widetilde{G})$ moves either vertically, or horizontally. Indeed, this follows directly from the assumption that G is a transform of F by a predictable sequence with values in $\{b,B\}$. This property combines nicely with biconvex functions: if ζ is such a function, then a successive application of Jensen's inequality gives

$$\mathbb{E}\zeta(\widetilde{F}_n,\widetilde{G}_n) \geq \mathbb{E}\zeta(\widetilde{F}_{n-1},\widetilde{G}_{n-1}) \geq \ldots \geq \mathbb{E}\zeta(\widetilde{F}_0,\widetilde{G}_0) = \zeta(0,0). \tag{3.8}$$

The distribution of the terminal variable $(\widetilde{F}_\infty,\widetilde{G}_\infty)$ gives rise to a probability measure ν on $\mathbb{R}^{2\times 2}_{sym}$: put

$$\nu\left(\mathrm{diag}(x,y)\right) = \mathbb{P}\big((\widetilde{F}_\infty,\widetilde{G}_\infty) = (x,y)\big), \qquad (x,y) \in \mathbb{R}^2,$$

where $\mathrm{diag}(x,y)$ stands for the diagonal matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Observe that ν is a laminate of barycenter 0. Indeed, if $\psi : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is a rank-one convex, then $(x,y) \mapsto \psi(\mathrm{diag}(x,y))$ is biconvex and thus, by (3.8),

$$\int_{\mathbb{R}^{2\times 2}} \psi \mathrm{d}\nu = \mathbb{E}\psi(\mathrm{diag}(\widetilde{F}_\infty,\widetilde{G}_\infty)) \geq \psi(\mathrm{diag}(0,0)) = \psi(\bar{\nu}).$$

Here we used the fact that $(\widetilde{F},\widetilde{G})$ is finite, so $(\widetilde{F}_\infty,\widetilde{G}_\infty) = (\widetilde{F}_n,\widetilde{G}_n)$ for some n.

Sharpness for Fourier multipliers, dimension $d = 2$. Recall the function $\Theta : [0,\infty) \to [0,1]$ which appears in (3.7). Then the function $\varphi : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}$ given by

$$\varphi(A) = \Theta\left(|BA_{11} + bA_{22}|\right) - (C^p_{p,b,B} - \varepsilon)\,|A_{11} + A_{22}|^p$$

is continuous. Hence, by Corollary 3.8, there is a functional sequence $(u_j)_{j\geq 1} \subset C_0^\infty(\mathbb{D})$ such that

$$\begin{aligned}\frac{1}{|\mathbb{D}|}\int_{\mathbb{R}^2} \varphi(D^2 u_j)\mathrm{d}x &= \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \varphi(D^2 u_j)\mathrm{d}x \\ &\xrightarrow{j\to\infty} \int_{\mathbb{R}^{2\times 2}_{sym}} \varphi \mathrm{d}\nu = \mathbb{E}\Theta(|G_\infty|) - (C^p_{p,b,B} - \varepsilon)\mathbb{E}\Theta(|F_\infty|) > 0.\end{aligned}$$

Therefore, for sufficiently large j, we have

$$\int_{\mathbb{R}^2} \Theta\left(\left|B\frac{\partial^2 u_j}{\partial x^2} + b\frac{\partial^2 u_j}{\partial y^2}\right|\right)\mathrm{d}x > (C^p_{p,b,B} - \varepsilon)\int_{\mathbb{R}^2} |\Delta u_j|^p \mathrm{d}x,$$

that, setting $f = \Delta u_j$,

$$\int_{\mathbb{R}^2} \Theta\left(\left|B\mathcal{R}_1^2 f + b\mathcal{R}_2^2 f\right|\right)\mathrm{d}x > (C^p_{p,b,B} - \varepsilon)\int_{\mathbb{R}^2} |f|^p \mathrm{d}x.$$

But $\chi_{[1,\infty)} \geq \Theta$, so we obtain

$$\left|\left\{x \in \mathbb{R}^2 : |T_m f(x)| \geq 1\right\}\right| > (C_{p,b,B}^p - \varepsilon) \int_{\mathbb{R}^2} |f|^p \mathrm{d}x,$$

with $T_m = B\mathcal{R}_1^2 + b\mathcal{R}_2^2$. This is the desired sharpness, since T_m is a multiplier with the symbol from the class (1.1).

Sharpness for Fourier multipliers, dimension $d \geq 3$. Suppose that for a fixed $p > 2$ and some positive constant C we have

$$\left|\left\{x \in \mathbb{R}^d : |BR_1^2 f(x) + bR_2^2 f(x)| \geq 1\right\}\right| \leq C\|f\|_{L^p(\mathbb{R}^d)}^p \tag{3.9}$$

for all f. For $t > 0$, define the dilation operator δ_t as follows: for any function $g : \mathbb{R}^2 \times \mathbb{R}^{d-2} \to \mathbb{R}$, we let $\delta_t g(\xi, \zeta) = g(\xi, t\zeta)$. We check that the operator $T_t := \delta_t^{-1} \circ (BR_1^2 + bR_2^2) \circ \delta_t$ satisfies

$$\begin{aligned}
&|\{x \in \mathbb{R}^d : T_t f(x) \geq 1\}| \qquad (3.10)\\
&= t^{d-2} \left|\left\{x \in \mathbb{R}^d : \left|\left(BR_1^2 + bR_2^2\right) \circ \delta_t f(x)\right| \geq 1\right\}\right|\\
&\leq Ct^{d-2} \int_{\mathbb{R}^d} |\delta_t f(x)|^p \mathrm{d}x = C\|f\|_{L^p(\mathbb{R}^d)}^p. \qquad (3.11)
\end{aligned}$$

Next, we easily check that the Fourier transform $\mathcal{F}$ satisfies the identity $\mathcal{F} = t^{d-2}\delta_t \circ \mathcal{F} \circ \delta_t$, so the operator T_t enjoys the condition

$$\widehat{T_t f}(\xi, \zeta) = -\frac{B\xi_1^2 + b\xi_2^2}{|\xi|^2 + t^2|\zeta|^2} \widehat{f}(\xi, \zeta), \qquad (\xi, \zeta) \in \mathbb{R}^2 \times \mathbb{R}^{d-2}.$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to 0} \widehat{T_t f}(\xi, \zeta) = \widehat{T_0 f}(\xi, \zeta)$$

in $L^2(\mathbb{R}^d)$, where $\widehat{T_0 f}(\xi, \zeta) = \frac{B\xi_2^2 + b\xi_1^2}{|\xi|^2} \widehat{f}(\xi, \zeta)$. By Plancherel's theorem and Fatou's lemma, we see that (3.11) implies

$$|\{x \in \mathbb{R}^d : |T_0 f(x)| > 1\}| \leq C\|f\|_{L^p(\mathbb{R}^d)}^p. \tag{3.12}$$

Now pick an arbitrary function $\tilde{f} \in L^p(\mathbb{R}^2)$ and set

$$f(\xi, \zeta) = \tilde{f}(\xi)\chi_{[0,1]^{d-2}}(\zeta).$$

Recalling that $\mathcal{R}_1$ and $\mathcal{R}_2$ are the *planar* Riesz transforms, we see that

$$T_0 f(\xi, \zeta) = (B\mathcal{R}_1^2 + b\mathcal{R}_2^2)\tilde{f}(\xi)\chi_{[0,1]^{d-2}}(\zeta),$$

because of the identity

$$\widehat{T_0 f}(\xi, \zeta) = -\frac{B\xi_2^2 + b\xi_1^2}{|\xi|^2} \widehat{\tilde{f}}(\xi)\widehat{1_{[0,1]^{d-2}}}(\zeta).$$

Plug this into (3.12) to obtain

$$\left|\left\{x \in \mathbb{R}^2 : \left|B\mathcal{R}_1^2\tilde{f} + b\mathcal{R}_2^2\tilde{f}\right| > 1\right\}\right| \leq C\|\tilde{f}\|_{L^p(\mathbb{R}^2)}^p.$$

However, we have shown above that this implies $C \geq C_{p,b,B}$. The proof is complete.

4. Further applications of Theorem 2.2: inequalities for harmonic functions and the Riesz system

4.1. **Inequalities for harmonic functions.** We will prove a version of Theorem 2.1 in the context of harmonic functions on Euclidean domains. Suppose that n is a positive integer and let D be an open connected subset of $\mathbb{R}^n$. Fix a base point ξ belonging to D and let $b \leq 0 < B$ with $b + B > 0$ be fixed real numbers. In addition, assume that two real-valued harmonic functions u, v on D satisfy the following conditions

$$\left|v(\xi) - \frac{B+b}{2}u(\xi)\right| \leq \left|\frac{B-b}{2}u(\xi)\right| \tag{4.1}$$

and

$$\left|\nabla v(x) - \frac{B+b}{2}\nabla u(x)\right| \leq \left|\frac{B-b}{2}\nabla u(x)\right| \quad \text{for any } x \in D. \tag{4.2}$$

Let D_0 be a bounded domain satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$ and let $\mu^\xi_{D_0}$ stand for the harmonic measure on ∂D_0 corresponding to ξ. The L^p norm of the function u is given by

$$||u||_{L^p(D)} = \sup_{D_0} \left(\int_{\partial D_0} |u|^p\right)^{1/p} \mathrm{d}\mu^\xi_{D_0},$$

where u_{D_0} is the restriction of u to D_0. The harmonic analogue of Theorem 2.1 is the following.

Theorem 4.1. *Let $b \leq 0 < B$ with $b + B > 0$ be fixed numbers. If u, v satisfy (4.1) and (4.2), then*

$$\|v\|_{p,\infty} \leq C_{p,b,B}\|u\|_p. \tag{4.3}$$

Proof. Let D_0 be an arbitrary subdomain of D as above. Let $W = (W_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^n$, started at ξ and stopped upon reaching the boundary of D_0. Then the processes $X = (u(W_t))_{t\geq 0}$ and $Y = (v(W_t))_{t\geq 0}$ are martingales and Itô's formula implies

$$X_t = u(\xi) + \int_0^t \nabla u(W_s)\mathrm{d}W_s,$$
$$Y_t - \frac{B+b}{2}X_t = v(\xi) - \frac{B+b}{2}u(\xi) + \int_0^t \left(\nabla v(W_s) - \frac{B+b}{2}\nabla u(W_s)\right)\mathrm{d}W_s.$$

Therefore, by (4.1) and (4.2), the process $Y - \frac{B+b}{2}X$ is differentially subordinate to $\frac{B-b}{2}X$, so for any $t \geq 0$ and $\lambda > 0$,

$$\lambda^p \mathbb{P}(|Y_t| \geq \lambda) \leq C^p_{p,b,B}\mathbb{E}|X_t|^p.$$

Since D_0 is bounded, W converges pointwise to the random variable which is distributed along ∂D_0 according to the measure $\mu^\xi_{D_0}$. Consequently, the right-hand side above tends to $\|u\|^p_{L^p(D_0)}$ as $t \to \infty$. The left-hand side is dealt with similarly and the claim follows, since D_0 was arbitrary. $\square$

4.2. **Inequalities for the Riesz system.** Let $w_0, w_1, \ldots, w_n$ be harmonic functions given on a domain $D \subset \mathbb{R}^{n+1}$ which consists of points of the form $x = (x_0, x_1, \ldots, x_n)$. Assume that the functions w_k have their values in the Hilbert space $\mathcal{H}$ and satisfy the generalized Cauchy-Riemann equations:

$$\sum_{k=0}^n w_{kk} = 0 \qquad \text{and} \qquad w_{jk} = w_{kj},$$

where $w_{jk} = \partial w_j/\partial x_k$, for $j, k = 0, 1, 2, \ldots, n$. If $w : D \to \mathcal{H}$ is harmonic, then $w_0 = \partial w/\partial x_0, \ldots, w_n = \partial w/\partial x_n$ satisfy these equations. As proved by Stein and Weiss [41], these systems of harmonic functions provide a natural setup for the extension of the theory of Hardy spaces to higher dimensions. Let $F = (0, w_1, \ldots, w_n)$ and $G = \frac{B+b}{2}(w_0, w_1, \ldots, w_n)$. Note that F and G are harmonic functions from D to $\mathcal{K} = \mathcal{H}\times\ldots\times\mathcal{H}$ where the norm of $y = (y_0, \ldots, y_n) \in \mathcal{K}$ is given by $\|y\|_{\mathcal{K}} = (\sum_{k=0}^n |y_k|^2)^{1/2}$. Then F and G satisfy

$$\left|\nabla G - \frac{B+b}{2}\nabla F\right| \leq \sqrt{n}\left|\frac{B-b}{2}\nabla F\right|,$$

such that b and B are as above. Indeed,

$$\begin{aligned}
\left|\nabla\left(G-\frac{B+b}{2}F\right)\right|^2 &= \left(\frac{B+b}{2}\right)^2|\nabla w_0|^2\\
&= \left(\frac{B+b}{2}\right)^2\left[|w_{00}|^2+\sum_{k=1}^n|w_{0k}|^2\right]\\
&= \left(\frac{B+b}{2}\right)^2\left[\left|\sum_{k=1}^n w_{kk}\right|^2+\sum_{k=1}^n|w_{k0}|^2\right]\\
&\le \left(\frac{B+b}{2}\right)^2\left[(n-1)\sum_{k=1}^n|w_{kk}|^2+\left(\sum_{k=1}^n|w_{kk}|^2+\sum_{k=1}^n|w_{k0}|^2\right)\right]\\
&\le \left(\frac{B+b}{2}\right)^2\left[(n-1)|\nabla F|^2+|\nabla F|^2\right]\\
&= n\left(\frac{B+b}{2}\right)^2|\nabla F|^2\\
&\le n\left(\frac{B-b}{2}\right)^2|\nabla F|^2.
\end{aligned}$$

Hence, by Theorem 4.1, if there is a point $\xi\in D$ such that

$$\left|G(\xi)-\frac{B+b}{2}F(\xi)\right|\le\sqrt{n}\left|\frac{B-b}{2}F(\xi)\right|,$$

then, for $1\le p<\infty$, we have

$$\|G\|_{p,\infty}\le\sqrt{n}C_{p,b,B}\|F\|_p.$$

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